07-04-16

• Ιδιότητες των εκτιμητών Ελαχίστων Τετραγώνων

$W = \sum a_i W_i$, $E(W) = \sum a_i E(W_i)$, $Var(W_i) = \sum a_i^2 Var(W_i)$

για ανεξάρτητες τυχαίες μεταβλητές W_i

$$\hat{\beta}_1 = \frac{\sum (x_i-\bar{x})(Y_i-\bar{Y})}{\sum (x_i-\bar{x})^2} = \frac{\sum (x_i-\bar{x})Y_i}{\sum (x_i-\bar{x})^2} - \cancelto{0}{\frac{\sum (x_i-\bar{x})\bar{Y}}{\sum (x_i-\bar{x})^2}}$$

$$\left\{ \begin{array}{l} \sum (x_i-\bar{x})\bar{Y} = 0 \\ \sum (x_i-\bar{x})\bar{X} = 0 \end{array} \right.$$

$$\Rightarrow E(\hat{\beta}_1) = \frac{\sum (x_i-\bar{x})}{\sum (x_i-\bar{x})^2} E(Y_i) =$$

$$= \cancelto{0}{\frac{\sum (x_i-\bar{x})\beta_0}{\sum (\ldots)^2}} + \frac{\sum (x_i-\bar{x})\beta_1 x_i}{\sum (\ldots)^2} =$$

$$= \beta_1 \frac{\sum (x_i-\bar{x})x_i}{\sum (\ldots)^2} = \beta_1 \left[\frac{\sum (x_i-\bar{x})x_i}{\sum (\ldots)^2} - \frac{\sum (x_i-\bar{x})\bar{x}}{\sum (\ldots)^2} \right] =$$

$$= \beta_1 \left[\frac{\sum (\ldots)^2}{\sum (\ldots)^2} \right] = \beta_1$$

$$\bullet\ Var(\hat{\beta}_1) = \sum \frac{(x_i-\bar{x})}{[(x_i-\bar{x})^2]^2} Var(Y_i) = \sigma^2 \frac{\sum_i^n (x_i-\bar{x})^2}{\left[\sum (x_i-\bar{x})^2\right]^2} = \frac{\sigma^2}{\sum (x_i-\bar{x})^2}$$

$$\bullet\ \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = \bar{Y} - \frac{\sum (x_i-\bar{x})Y_i \bar{x}}{\sum (x_i-\bar{x})^2}$$

$$= \sum \left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{\sum (x_i-\bar{x})^2} \right) Y_i \rightsquigarrow E(\hat{\beta}_0) = \sum_{i=1}^{n} \left(\frac{1}{n} \; \frac{\bar{x}(x_i-\bar{x})}{\sum (x_i-\bar{x})^2} \right)(\beta_0 + \beta_1 x_i) =$$

$$= \beta_0 \sum\left(\frac{1}{n} \frac{\bar{x}(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}\right) + \beta_1 \sum\left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}\right) x_i =$$

$$= \beta_0\left(\frac{n}{n} - \frac{\bar{x}\overbrace{\sum(x_i-\bar{x})}^{\to 0}}{\sum(\dots)^2}\right) + \beta_1\left(\frac{1}{n}\sum x_i - \frac{\bar{x}\sum(x_i-\bar{x})x_i}{\sum(\dots)^2}\right)$$

$$= \beta_0 + \beta_1\left(\bar{x} - \bar{x}\,\frac{\left[\sum(x_i-\bar{x})x_i - \sum(x_i-\bar{x})\bar{x}\right]}{\sum(x_i-\bar{x})^2}\right) =$$

$$= \beta_0 + \beta_1\left(\bar{x} - \bar{x}\,\frac{\sum(\dots)^2}{\sum(\dots)^2}\right) = \beta_0$$

$$Var(\hat{\beta}_0) = \sum_{i=1}^{n}\left(\frac{1}{n} - \frac{\bar{x}(x_i-\bar{x})}{\sum(\dots)^2}\right)^2 Var(Y_i) =$$

$$= \sigma^2 \sum_{i=1}^{n}\left(\frac{1}{n^2} + \frac{\bar{x}^2(x_i-\bar{x})^2}{\left[\sum(x_i-\bar{x})^2\right]^2} - \frac{2}{n}\,\bar{x}\,\frac{(x_i-\bar{x})}{\sum(x_i-\bar{x})^2}\right]$$

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2\sum(x_i-\bar{x})^2}{\left[\sum(x_i-\bar{x})^2\right]^2} - \frac{2}{n}\,\frac{\sum\bar{x}(x_i-\bar{x})}{\sum(x_i-\bar{x})^2} =\right.$$

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}}{\sum(x_i-\bar{x})^2}\right)$$

$$\hat{\sigma}^2 = S^2 = \frac{1}{n-2}\sum_{1}^{n} e_i^2 = \frac{1}{n-2}\sum_{1}^{n}\left(Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i\right)^2$$

• Στατιστική Συμπερασματολογία

$\varepsilon_i \sim N(0, \sigma^2) \longrightarrow Y_i \sim N(E(Y_i) = (\beta_0 + \beta_1 x_i, \sigma^2)$

$\frac{(n-2) S^2}{\sigma^2} \sim X^2_{n-2}$

$\hat{\beta}_0 \sim N(\beta_0, \sigma^2 (\frac{1}{n} + \frac{\bar{x}^2}{\Sigma(x_i - \bar{x})^2})$

$\hat{\beta}_1 \sim N\left(\beta_1, \frac{\sigma^2}{\Sigma(x_i - \bar{x})^2}\right)$

$$\frac{\hat{\beta}_1 - \beta_1}{\sigma / \sqrt{\Sigma(x_i - \bar{x})^2}} \sim N(0,1)$$

$$\frac{\quad}{\sqrt{\frac{(n-2)S^2}{\sigma^2} / (n-2)} \sim (X^2_{n-2}/n-2)} \equiv t_{n-2}$$

$$\beta_0 = \frac{\hat{b}_0 - b_0}{S\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\Sigma(x_i - \bar{x})^2}}} \sim t_{n-2}$$

$$\beta_1 = \frac{\hat{b}_1 - b_1}{S / \sqrt{\Sigma(x_i - \bar{x})^2}} \sim t_{n-2}$$

$(1-\alpha)100\%$ ΔΕ για το β_0:

$$\hat{\beta}_0 \pm t_{\alpha/2, n-2} \cdot S\sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum(x_i-\bar{x})^2}}$$

$$\beta_1 : \hat{\beta}_1 \pm t_{\alpha/2, n-2} : \frac{S}{\sqrt{\sum(x_i-\bar{x})^2}}$$

$\rightarrow \sum(x_i-\bar{x})^2 = 3.400$

$\rightarrow S^2 = 60/8 = 7{,}5$

95% Δ.Ε για το β_1: 2.0 ± 2.306

$$\frac{\sqrt{7{,}5}}{\sqrt{3.400}} \rightsquigarrow \{1{,}89, 2.11\}$$

$\rightsquigarrow$ Έλεγχος της υπόθεσης

$H_0: \beta_1 = 0$ v $H_a: \beta_1 \neq 0$

Ο έλεγχος γίνεται με το B_1 (με $\beta_1 = 0$) $\sim t_{n-2}$, όταν H_0: αληθής και κρίσιμη περιοχή $|B_1| \geq t_{\alpha/2, n-2}$

• Διάστημα Εμπιστοσύνης για τη συνάρτηση παλινδρόμησης

$E(Y) = \beta_0 + \beta_1 x$ με $\hat{Y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0$ για $x = x_0$

$E(\hat{Y}_0) = \beta_0 + \beta_1 x_0 = E(Y_0)$

$Var(\hat{Y}_0) = \sigma^2 \left(\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{\Sigma (x_i - \bar{x})^2} \right)$ Δε χρειάζεται απόδειξη!

$\hat{Y}_0 \sim N(E(Y_0), Var(\hat{Y}_0))$ και $\frac{\hat{Y}_0 - E(Y_0)}{S\sqrt{\frac{1/n + (x_0 - \bar{x})^2}{\Sigma (x_i - \bar{x})^2}}} \sim t_{n-2}$

$(1-a)100\%$ ΔΕ για $E(Y_0)$:

$\hat{Y}_0 \pm t_{a/2, n-2} \cdot S \sqrt{\frac{1}{n} + \frac{(x - \bar{x})^2}{\Sigma (x_i - \bar{x})^2}}$

• 90% ΔΕ για $E(Y \mid X = 55)$: $120 \pm 1.860 * \sqrt{7{,}5} \sqrt{\frac{1}{10} + \frac{(55-50)}{\sqrt{3400}}} =$

$= \{118{,}3 \; , \; 121{,}7\}$

$\hat{Y}_0 = \hat{\beta}_0 + \hat{\beta}_1 x_0 = 10 + 2 * 55 = 120$, $t_{a/2, n-2} = t_{0.05, 8} = 1.860$

$\Sigma(\ldots)^2 = 3.400$, $\bar{x} = 50$, $S = \sqrt{7{,}5}$